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COMPOSITION SERIES FOR ANALYTIC CONTINUATIONS OF HOLOMORPHIC DISCRETE SERIES REPRESENTATIONS OF $SU(n, n)$

BY

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ABSTRACT. We study a certain family of holomorphic discrete series representations of the semisimple Lie group $G = SU(n, n)$ and the corresponding analytic continuation in the inducing parameter λ. At the values of λ where the representations become reducible, we compute the composition series in terms of a Peter-Weyl basis on the Shilov boundary of the Hermitian symmetric space for G.

Introduction. Holomorphic vector bundles E over a Hermitian symmetric space G/K have been studied extensively, in particular in connection with realization of the holomorphic discrete series for G. One knows in certain cases that by considering the Harish-Chandra module M for G consisting of K-finite sections of E, it is possible to obtain an analytic continuation in the parameters of the representation of K defining E, so that new (nondiscrete) unitary representations of G ensue [**5**]. These are bottom subquotients of M and defined by differential equations on the Shilov boundary. In this paper we will describe the full composition series for M in the special case of line bundles for the group $SU(n, n)$ and its universal covering. Our computation of the composition series is not based (as usual) on intertwining operators (partly because the subquotients seem to be new, mostly nonunitary, representations) but rather on the behavior of the G-invariant (nondefinite in general) Hermitian form on M. The basic formula is an expansion of the distribution $\det(1 - x)^{-\lambda}$ on $U(n)$ in terms of the Peter-Weyl L^2-basis.

In §1 we derive this "binominal formula" affording a determination of the invariant Hermitian form on M, and in §2 we derive the composition series, expressed in terms of a certain ordering of the holomorphic dual of $U(n)$. Finally in §3 we list a result and a conjecture relating the K-spectrum and the N-spectrum of the modules.

The author has benefited from conversations with H. P. Jakobsen, I. E. Segal, B. Speh (whose paper [**8**] gives a thorough treatment of the case of $SU(2, 2)$ as well as $SO(n, 2)$) and M. Vergne.

1. Let G denote the group $SU(n, n)$ [**1**] for a fixed n and K its maximal compact subgroup which is isomorphic to $U(1) \times SU(n) \times SU(n)$. The character

$$\chi_\lambda \colon e^{i\theta} \to e^{i\lambda n\theta} \tag{1}$$

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on the center of $\tilde{K}$, the universal covering of K, is a holomorphic character of K for λ a nonnegative integer and in this case determines a holomorphic line bundle E_λ on G/K. As a model for G/K we take the generalized unit disc

$$\mathbf{D} = \{Z \in M(n, \mathbf{C}) | I - Z^*Z > 0\}$$

i.e. all $n \times n$ matrices of operator norm strictly smaller than one. Elements g in G are written in terms of $n \times n$ blocks

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ where } A^*A - C^*C = I, A^*B = C^*D, B^*B - D^*D = -I$$

and the action is $g \cdot Z = (AZ + B)(CZ + D)^{-1}$. K consists of those elements k with $B = C = 0$ and $\chi_\lambda(k) = (\det A)^\lambda$ for $k \in K$ and λ an integer. We trivialize E_λ over G/K and get as sections holomorphic functions on $\mathbf{D}$ with the G-action

$$U_\lambda(g): f(Z) \to \det(CZ + D)^{-\lambda} f(g^{-1} \cdot Z), \qquad g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2}$$

For $\lambda \geqslant 2n$ this action is unitary with respect to the L^2-norm

$$\int_{\mathbf{D}} |f(Z)|^2 \det(I - Z^*Z)^{\lambda - 2n}\, dZ \tag{3}$$

where dZ is Lebesgue measure on $M(n, \mathbf{C})$. (2) gives the part of the holomorphic discrete series for G corresponding to a character of K [5]. Working on the unbounded (tube) realization of G/K, [5] described unitary irreducible (nondiscrete) representations of G corresponding to $\lambda = 0, 1, 2, \ldots, 2n - 1$. We will study the analytic continuation in λ of the above situation using instead the realization $\mathbf{D}$ of G/K and replace Fourier analysis over $H(n)$, all Hermitian $n \times n$ matrices, as in [5] by Fourier analysis over $U(n)$, all $n \times n$ unitary matrices. This approach seems more convenient when dealing with Harish-Chandra modules, and it also sheds some light on the connection between the K-spectrum and the N-spectrum of a representation.

Let M denote the vector space of all holomorphic polynomials on $\mathbf{D}$, i.e. polynomials in the coordinates of $Z \in \mathbf{D}$. M is a K-module under the action in (2) viz.

$$U_\lambda(k): p(Z) \to (\det D)^\lambda p(A^{-1}ZD), \qquad k = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \tag{4}$$

where λ is an integer (for a general λ, M is a module for $\tilde{K}$, the universal covering). M is also a module for $\mathfrak{G}$, the Lie algebra of G, by the infinitesimal representation corresponding to (2) which is

$$u_\lambda(X): p(Z) \to -\lambda \operatorname{tr}(cZ + d)p(Z) + (\tilde{X}p)(Z). \tag{5}$$

Here $\tilde{X}$ differentiates p (via the holomorphic chain rule say) in the direction of the infinitesimal action

$$X: Z \to aZ + b - ZcZ - Zd, \qquad -X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(4) and (5) defines (for each λ) M as a $(\mathfrak{G}, \tilde{K})$-module with finite $\tilde{K}$-multiplicities (a Harish-Chandra module), in fact the $\tilde{K}$-types have multiplicity one (which of course is in agreement with general results [6]).

Consider $U(n)$, the Shilov boundary of **D**, and the set P of all its holomorphic representations (i.e. its irreducible polynomial representations). These are labeled by n-tuples **[10]** of integers l: $0 \leqslant l_n < l_{n-1} < \cdots < l_1$ (or in the highest weight terminology $(f_n, f_{n-1}, \ldots, f_1)$ nondecreasing:

$$f_n = l_n, f_{n-1} = l_{n-1} - 1, \ldots, f_1 = l_1 - n + 1,$$

where l is derived from a Young diagram with f_1 boxes in the first row, f_2 in the second and so on). Let D^l denote the corresponding representation and $D^l_{ij}(Z)$ its matrix coefficients (polynomials on $M(n, \mathbf{C})$). Clearly, under (4), $D^l_{ij}(Z)$ transforms according to the representation $\chi_{-\lambda-2d(l)/n} \times D^l \times D^{l'}$ of $\tilde{K}$, where l' denotes the contragradient Young diagram and $d(l)$ the degree of the polynomial $D^l_{ij}(Z)$ (in short, call this $\tilde{K}$-type (λ, l), a module of dimension $N(l)^2$ where $N(l)$ is the dimension of D^l).

Recall next that for λ a discrete series parameter ($\lambda \geqslant 2n$), the Hilbert space $H(\lambda)$ defined by (3) (i.e. all holomorphic functions for which (3) is finite) has a reproducing kernel

$$K_\lambda(Z, W) = \det(I - W^*Z)^{-\lambda} \tag{6}$$

corresponding to a normalized measure

$$c(\lambda)\int_{\mathbf{D}} \det(I - Z^*Z)^{\lambda-2n}\, dZ. \tag{7}$$

Here in fact (as seen by a use of polar coordinates **[2]**)

$$c(\lambda) = c\frac{\Gamma(\lambda)\Gamma(\lambda - 1)\ldots\Gamma(\lambda - n + 1)}{\Gamma(\lambda - n)\Gamma(\lambda - n - 1)\ldots\Gamma(\lambda - 2n - 1)}$$

where c only depends on n.

For the discrete series, $H(\lambda)$ contains all polynomials in **D** which turns out to be precisely all polynomials of the form of finite linear combinations of the D^l_{ij}'s, $l \in P$. Hence we have for the K-finite vectors

$$H(\lambda)_K = \operatorname{span}\left\{D^l_{ij} | l \in P, i,j = 1, \ldots, N(l)\right\}. \tag{8}$$

(8) is one of the consequences of the following

PROPOSITION 1. *For each real* λ, K_λ *in* (6) *has the following infinite series representation, uniformly convergent on compact subsets of* **D** $\times$ **D**,

$$K_\lambda(Z, W) = \sum_{l \in P} \sum_{i,j=1}^{N(l)} a_l(\lambda) D^l_{ij}(Z)\, \overline{D^l_{ij}(W)} \tag{9}$$

where $a_l(\lambda)$ *is the meromorphic function* (*with removable singularities and zeroes of finite order*)

$$a_l(\lambda) = \frac{P(l_1 \ldots l_n)(\lambda + l_1 - n)! \ldots (\lambda + l_n - n)!}{l_1! \ldots l_n!\,(\lambda - 1)! \ldots (\lambda - n)!}.$$

Here we have introduced the difference product $P(l_1 \ldots l_n) = \Pi_{i<j}(l_i - l_j)$ *and the usual meromorphic continuation* $\lambda! = \Gamma(\lambda + 1)$ *of factorial.*

PROOF. The expansion in (9) also reads

$$K_\lambda(Z, W) = \sum_l a_l(\lambda) \operatorname{tr} D^l(W^* Z)$$

so we will put $Z = W$ (a holomorphic/antiholomorphic function in 2 variables is determined by its values on the diagonal). Consider now

$$f_\lambda(x) = \det(1 - x)^{-\lambda} \tag{10}$$

as a function (of one variable) on $U(n)$, the Shilov boundary of $\mathbf{D}$. For λ a negative integer this is a polynomial function on $U(n)$, in all other cases it is a distribution on $U(n)$, possibly singular along $\{x|\det(1 - x) = 0\}$.

Since (10) is invariant under inner automorphisms we can integrate it against characters of representations of $U(n)$ by integrating only over the maximal torus. Specifically [8], as before, let $N(l)$ be the dimension of D^l,

$$N(l) = P(l_1, \ldots, l_n)/P(n - 1, \ldots, 0).$$

The character of D^l is on the maximal torus

$$\chi_l(x) = |\varepsilon^{l_1} \ldots \varepsilon^{l_n}|/|\varepsilon^{n-1} \ldots \varepsilon^0|$$

where

$$x = \begin{bmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{bmatrix}, \qquad \varepsilon_j = e^{2\pi i \varphi_j}, 0 \leqslant \varphi_j < 1,$$

and $|\varepsilon^{l_1} \ldots \varepsilon^{l}| = \det(\varepsilon_i^{l_j})_{i,j=1}^n$. Now the integral over $U(n)$ of any invariant function $h(x)$ is

$$\int_{U(n)} h(y)\, dy = \frac{1}{n!} \int_0^1 \cdots \int_0^1 h(x)|d(x)|^2\, d\varphi_1 \ldots d\varphi_n$$

when

$$d(x) = |\varepsilon^{n-1} \ldots \varepsilon^0| \quad \text{for } x = \begin{bmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{bmatrix} \text{ as above.}$$

Also, here $\int_{U(n)} dy = 1$.

Hence

$$\int_{U(n)} f_\lambda(y) \overline{\chi_l(y)}\, dy = \frac{1}{n!} \int_0^1 \cdots \int_0^1 (1 - e^{2\pi i \varphi_1})^{-\lambda} \ldots (1 - e^{2\pi i \varphi_n})^{-\lambda}$$

$$\cdot \left|\overline{\varepsilon^{l_1} \ldots \varepsilon^{l_n}}\right| d(x)\, d\varphi_1 \ldots d\varphi_n \tag{11}$$

where again

$$|\varepsilon^{l_1} \ldots \varepsilon^{l_n}| = \sum_\rho (-1)^\rho e(l_1 \varphi_{\rho(1)} + \cdots + l_n \varphi_{\rho(n)}),$$

sums over all permutations of n letters, $(-1)^{\rho}$ the signature of ρ and $e(\theta) = e^{2\pi i\theta}$. We can find the Fourier series for $(1 - e(\theta))^{-\lambda}$ for all real λ as a distribution on S^1, viz.

$$(1 - e(\theta))^{-\lambda} = \sum_{j=0}^{\infty} (-e(\theta))^j \binom{-\lambda}{j} = \sum_{j=0}^{\infty} \frac{(\lambda + j - 1)!}{j!\,(\lambda - 1)!} e(\theta)^j. \tag{12}$$

In the same way $f_\lambda(y)$ is a distribution on $U(n)$, namely the unique meromorphic continuation of the cases λ a negative integer. Therefore, to evaluate (11) we insert (12) and use the usual orthogonality relations for $\{e(\theta)^j\}$ on S^1,

$$\begin{aligned} n! \int_{U(n)} f_\lambda(y) \overline{\chi_l(y)}\, dy &= \int_0^1 \cdots \int_0^1 \sum_{j_1=0}^{\infty} \frac{(\lambda + j_1 - 1)!}{j_1!\,(\lambda - 1)!} e(\varphi_1)^{j_1} \cdots \sum_{j_n=0}^{\infty} \frac{(\lambda + j_n - 1)!}{j_n!\,(\lambda - 1)!} e(\varphi_n)^{j_n} \\ &\quad \cdot \Big(\sum_{\rho} (-1)^{\rho} e(l_1 \varphi_{\rho(1)} + \cdots + l_n \varphi_{\rho(n)})^{-} \\ &\qquad \cdot \sum_{\pi} (-1)^{\pi} e((n-1)\varphi_{\pi(1)} + \cdots + 0 \cdot \varphi_{\pi(n)}) \Big)\, d\varphi_1 \ldots d\varphi_n. \end{aligned}$$

Bringing the sum over ρ outside the integrals, each of the $n!$ terms will be equal to

$$\begin{aligned} &\sum_{\pi} (-1)^{\pi} \frac{(\lambda + l_1 - \pi(n-1) - 1)!}{(l_1 - \pi(n-1))!\,(\lambda - 1)!} \cdots \frac{(\lambda + l_n - \pi(0) - 1)!}{(l_n - \pi(0))!\,(\lambda - 1)!} \\ &= \frac{P(l_1 \ldots l_n)}{l_1! \ldots l_n!} \frac{(\lambda + l_1 - n)! \ldots (\lambda + l_n - n)!}{(\lambda - 1)! \ldots (\lambda - n)!}. \end{aligned}$$

The last simplification has as an intermediate step the expression

$$\begin{aligned} &\left(\frac{1}{(\lambda - 1)!} \right)^n \frac{(\lambda + l_1 - n)! \ldots (\lambda + l_n - n)!}{l_1! \ldots l_n!} \\ &\cdot \sum_{\pi} (-1)^{\pi} \big[l_1(l_1 - 1) \ldots (l_1 - \pi(n-1) - 1)(\lambda + l_1 - \pi(n-1) - 1) \\ &\qquad \cdot (\lambda + l_1 - \pi(n-1) - 2) \ldots (\lambda + l_1 - n + 1) \\ &\qquad \cdot l_2(l_2 - 1) \ldots (l_2 - \pi(n-2) - 1)(\lambda + l_2 - \pi(n-2) - 1) \\ &\qquad \cdot (\lambda + l_2 - \pi(n-2) - 2) \ldots (\lambda + l_2 - n + 1) \\ &\qquad \vdots \\ &\qquad \cdot l_n(l_n - 1) \ldots (l_n - \pi(0) - 1)(\lambda + l_n - \pi(0) - 1) \\ &\qquad\qquad \cdot (\lambda + l_n - \pi(0) - 2) \ldots (\lambda + l_n - n + 1) \big] \end{aligned}$$

where the sum over π is equal to $P(l_1 \ldots l_n) q(\lambda)$ and $q(\lambda)$ is the polynomial of degree $\frac{1}{2} n(n-1)$,

$$q(\lambda) = (\lambda - 1)^{n-1} (\lambda - 2)^{n-2} (\lambda - 3)^{n-3} \ldots (\lambda - n + 1).$$

If we now consider the analytic continuation to $\mathbf{D}$ of f_λ we have shown that

$$\det(I - Z)^{-\lambda} = \sum_l a_l(\lambda)\chi_l(Z)$$

$$\text{where } a_l(\lambda) = \frac{P(l_1 \dots l_n)}{l_1! \dots l_n!} \frac{(\lambda + l_1 - n)! \dots (\lambda + l_n - n)!}{(\lambda - 1)! \dots (\lambda - n)!} \tag{13}$$

is meromorphic in λ and has generically zeroes of finite order at $\lambda = n - 1, n - 2, n - 3, \dots$. The series in (13) is finite for λ a negative integer and converges uniformly on compact subsets of $\mathbf{D}$ for all real values of λ. This gives us the desired expansion (9) since

$$\chi_l(W^*Z) = \sum_{i,j=1}^{N(\lambda)} D_{ij}^l(Z)\overline{D_{ij}^l(W)}. \quad \square$$

REMARK 2. As later pointed out to us by R. Stanley there is a combinatorial formula [7],

$$\sum_l \chi_l(x)\chi_l(y) = \prod_{i,j}(1 - x_i y_j)^{-1},$$

the sum being over all Young diagrams (any number of rows). This is a formal identity with x and y (infinite) diagonal matrices. By choosing

$$x = (x_1, x_2, \dots, x_n, 0, 0, \dots), \qquad y = (1, 1, \dots, 1, 0, 0, \dots)$$

with m 1's, we get formally

$$\sum_l \chi_l(x)\chi_l(1) = \prod_{i=1}^{n}(1 - x_i)^{-m}$$

where the sum is over diagrams of at most $\min(m, n)$ rows. It is conceivable that one can show that $\chi_l(1)$ as a function of m has a meromorphic continuation to the form given in (9) above. At any rate, this argument certainly demonstrates that for $\lambda = 0, 1, \dots, n - 1$, the sum in (9) is only over Young diagrams with at most λ rows (in agreement with our results).

Note that for $\lambda = 2n$ the Hilbert space $H(2n)$ is just all holomorphic functions, square-integrable with respect to Lebesgue measure, and (9) is just the Bergman kernel expanded in terms of the $D_{ij}^l(Z)$,

$$\det(I - W^*Z)^{-2n} = \sum_{l,i,j} a_l(2n)D_{ij}^l(Z)\overline{D_{ij}^l(W)}. \tag{14}$$

On the other hand, the Bergman kernel always has an expansion [1] in terms of an orthonormal basis $\{\psi_m\}_{m=1}^{\infty}$ of $H(2n)$,

$$\det(I - W^*Z)^{-2n} = \sum_{m=1}^{\infty} \psi_m(Z)\overline{\psi_m(W)}$$

where the sum again is uniformly convergent on compact subsets. By general results on unitary representations of compact groups the sum in (14) is over mutually orthogonal representations of K, so we conclude that, in $H(2n)$, $\{a_l(2n)^{1/2}D_{ij}^l(Z) \mid l, i, j\}$ is an orthonormal basis. In particular (14) is the usual expansion of the Bergman kernel in terms of a basis and $\{D_{ij}^l \mid l, i, j\}$ is a basis for all polynomials on **D**. This last statement has the combinatorial corollary that the number of independent polynomials in n^2 variables of degree d is equal to $\Sigma_{d(l)=d}N(l)^2$ where again $N(l) = \dim D^l$ and $d(l)$ the degree, $d(l) = l_1 + \cdots + l_n - \frac{1}{2}n(n-1)$.

Note also that for $\lambda = n$, the expansion (9) is still a sum over all l, and that $a_l(n) = N(l)$ corresponding to the fact that $\int_{U(n)}|D^l_{ij}|^2 = N(l)^{-1}$. Indeed as we shall see (see also [5]) $\lambda = n$ corresponds to the Hardy-space representation over the Shilov boundary $U(n)$.

PROPOSITION 3. *For $\lambda \geqslant 2n$ the Hilbert space $H(\lambda)$ has the orthonormal basis $\{a_l(\lambda)^{1/2}D_{ij}(Z)|l, i, j\}$ and its reproducing kernel has the expansion* (9) *analogous to the case $\lambda = 2n$ in terms of this basis.*

The representation (2) of $\tilde{G}$ is unitary and irreducible in $H(\lambda)$ and so is, by Harish-Chandra's theorem, the infinitesimal $(\mathfrak{G}, \tilde{K})$ module M of all polynomials on **D**.

2. Let M be the $(\mathfrak{G}, \tilde{K})$ module as above with the actions (4) and (5) for each λ. From the previous section we have $M = \operatorname{span}\{D^l_{ij}(Z)|l, i, j\}$ and for $\lambda \geqslant 2n$ the Hermitian form (positive for $\lambda \geqslant 2n$)

$$\left(D^l_{ij}, D^{l'}_{i'j'}\right)_\lambda = a_l(\lambda)^{-1}\delta_{ii'}\delta_{jj'}\delta_{ll'} \tag{15}$$

is invariant under $\mathfrak{G}$, viz.

$$(u_\lambda(X)v, w)_\lambda = -(v, u_\lambda(X)w)_\lambda \qquad (X \in \mathfrak{G}, v, w \in M). \tag{16}$$

Any irreducible Harish-Chandra module has defined on it a unique invariant Hermitian form (possibly nondefinite). In our case we can give the form directly on the indecomposable (possibly reducible for certain λ) module M by simply continuing (15) analytically in λ. For fixed $v, w \in M$, $(v, w)_\lambda$ is then a meromorphic function of λ, and by analytic continuation of (16) we get that for fixed $v, w \in M$ and $X \in \mathfrak{G}$, the meromorphic functions $(u_\lambda(X)v, w)_\lambda$ and $-(v, u_\lambda(X)w)_\lambda$ coincide. In this sense we have for all λ an invariant Hermitian form on M, and it turns out that reducibility of M exactly corresponds to the existence of singularities of $(v, v)_\lambda$ for some $v \in M$.

PROPOSITION 4. *For $\lambda = 0, 1, \ldots, n-1$ (and actions as in* (4) *and* (5)) *M has a composition series (sharp inclusions)*

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M \tag{17}$$

of length $m = n - \lambda$. M_0 is the K-finite vectors for the unitary irreducible representation given in [5] *and consists of Young diagrams of at most λ rows (i.e.*

$$M_0 = \operatorname{span}\left\{D^l_{ij}|l\colon 0 < 1 < \cdots < n - \lambda - 1 < l_\lambda < \cdots < l_1\right\}\Big).$$

PROOF. By cancellation of the poles in the numerator and denominator, $a_l(\lambda)^{-1}$ is finite for $[l] \leqslant \lambda$ (at most λ rows) whereas in all other cases it is infinite (a pole of order greater than or equal to 1 and smaller than or equal to $n - \lambda$). Even in this singular case, the Hermitian form is invariant so that M_0 must be an invariant subspace of M. On the other hand, certain elements in the complexification of $\mathfrak{p}$, the reductive complement of the Lie algebra of K, act on M by raising or lowering the weights of the compact Cartan subgroup (sending a $\tilde{K}$-isotypic component into a higher one). The invariant form is uniquely determined by the action of $\mathfrak{p}$ so that the only way reducibility can occur is when the analytic continuation of the Hermitian form has a pole. In fact using the same argument for the form $p(\lambda')(v, w)_{\lambda}$, where $p(\lambda')$ is a polynomial with a zero of finite order at $\lambda' = \lambda$ (the first $p(\lambda') = \lambda' - \lambda$) one sees that the composition series in (17) indeed has irreducible quotients, and that M_i consists of those vectors v for which $(v, v)_{\lambda'}$ has a pole of order i at $\lambda' = \lambda$. □

REMARK 5. The unitarizable M_0 has as its completion the reproducing kernel Hilbert space $H(\lambda)$ generated by $\det(I - W^*Z)^{-\lambda}$ [5].

From the proof of the proposition we have

COROLLARY 6. *For $\lambda = 0, 1, 2, \ldots, n-1$ the subquotients in* (17) *are nonunitary except for M_0 and M/M_{m-1} which is isomorphic to the holomorphic nondiscrete series module $H(\lambda')$ corresponding to the parameter $\lambda' = 2n - \lambda$.*

REMARK 7. From what is just said, we also have Proposition 3 for $n < \lambda < 2n$, in which case the inner product is a Hardy-type norm as explained in [5]. The $H(\lambda')$ in Corollary 6 is then just the completion of M with respect to $(\cdot, \cdot)_{\lambda'}$.

PROOF. Consider first M/M_0 with the induced invariant form $(\lambda' - \lambda)(\cdot, \cdot)_{\lambda'}|_{\lambda'=\lambda}$ so that in M/M_0, D^l_{ij} has "norm-squared" equal to

$$\frac{l_1! \ldots l_n!}{P(l_1 \ldots l_n)} \frac{(\lambda-1)! \ldots (\lambda-\lambda)!}{(\lambda+l_1-n)! \ldots (\lambda+l_\lambda-n)!} \frac{(-1)!}{(\lambda+l_{\lambda+1}-n)!} \frac{(-2)!}{(\lambda+l_{\lambda+2}-n)!} \cdots \frac{(\lambda-n)!}{(\lambda+l_n-n)!}(\lambda-\lambda)$$

where (with a slight abuse of notation)

$$(-1)!\,(\lambda-\lambda) = (\lambda'-\lambda-1)!\,(\lambda'-\lambda)|_{\lambda'=\lambda} = -1.$$

and the expression is finite provided e.g.

$$l_n = 0, l_{n-1} = 1, \ldots, l_{\lambda+2} = n-\lambda-2, l_{\lambda+1} > n-\lambda-1.$$

To see a little more clearly what happens, let us consider the case $n = 4$ and write the "norm-squared" of D^l_{ij} for various l (and again the slight abuse

$$(-1)! = \frac{\text{Res } \lambda'!}{\lambda'-\lambda}\Big|_{\lambda'=\lambda}\Big)$$

omitting $l_1! \ldots l_n!/P(l_1 \ldots l_n)$.

$$\lambda = 3: l: l_1l_2l_3 0: \quad \frac{2!1!0!}{(l_1-1)!\,(l_2-1)!\,(l_3-1)!} \quad \text{(finite)}$$

$$l_1l_2l_3l_4: \quad \frac{2!1!0!\,(-1)!}{(l_1-1)!\,(l_2-1)!\,(l_3-1)!\,(l_4-1)!}$$

$$\lambda = 2:\quad l_1 l_2 10:\quad \frac{1!0!}{(l_1-2)!\,(l_2-2)!} \qquad \text{(finite)}$$

$$l_1 l_2 l_3 0:\quad \frac{1!0!\,(-1)!}{(l_1-2)!\,(l_2-2)!\,(l_3-2)!}$$

$$l_1 l_2 l_3 l_4:\quad \frac{1!0!\,(-1)!\,(-2)!}{(l_1-2)!\,(l_2-2)!\,(l_3-2)!\,(l_4-2)!} \qquad \text{etc.}$$

In this case the composition series is defined by

$\lambda = 3$:	M_0: 1 l_i	less than	1,
$\lambda = 2$:	M_0: 2 l_i's	– –	2,
	M_1: 1 l_i	– –	2,
$\lambda = 1$:	M_0: 3 l_i's	– –	3,
	M_1: 2 l_i's	– –	3,
	M_2: 1 l_i	– –	3.

More generally for any n we have

$\lambda = n-1$:	M_0: 1	l_i	less than	1,
$\lambda = n-2$:	M_0: 2	–	– –	2,
	M_1: 1	–	– –	2,
$\lambda = n-3$:	M_0: 3	–	– –	3,
	M_1: 2	–	– –	3,
	M_2: 1	–	– –	3,
$\lambda = n-4$:	M_0: 4	–	– –	4,
	M_1: 3	–	– –	4,
	M_2: 2	–	– –	4,
	M_3: 1	–	– –	4,
⋮				
$\lambda = 1$:	M_0: $n-1$	–	– –	$n-1$,
	M_1: $n-2$	–	– –	$n-1$,
	M_2: $n-3$	–	– –	$n-1$,
	⋮			
	M_{n-2} 1	–	– –	$n-1$

where all intermediate quotients are irreducible but nonunitary. The nonunitary follows from the fact that the Γ-function has residue $(-1)^m$ at $x = -m$ so that the sign of the inner product changes within the same subquotient. The top subquotient however is unitary and has a 1-dimensional highest weight vector (like the remaining modules), namely just a certain power of det Z. The action of the center of $\tilde{K}$ identifies this quotient as the K-finite vectors of $H(\lambda')$, $\lambda' = 2n - \lambda$ (in fact K itself acts here). Note that $H(\lambda')$ is a well-defined reproducing kernel Hilbert space by our binomial formula (9) (and [5]). □

The same line of reasoning is present in the proof of the following

PROPOSITION 8. *M is an irreducible module exactly for $\lambda \neq n-1, n-2, n-3, \ldots,$ and for λ a negative integer it has a composition series* (17) *of length $m = n$. In that case the bottom subquotient is finite dimensional and the top subquotient is a holomorphic discrete series with parameter $\lambda' = 2n - \lambda$. M_0 is unitary precisely for* $\lambda \in \{0, 1, \ldots, n-1\} \cup [n, \infty)$.

PROOF. Again the composition series comes about via

$$\begin{array}{lllr} M_0: & n & l_i\text{'s} & \text{less than } n-\lambda, \\ M_1: & n-1 & - & - \; - \; n-\lambda, \\ M_2: & n-2 & - & - \; - \; n-\lambda, \\ & \vdots & & \\ M_{n-1}: & 1 & - & - \; - \; n-\lambda \end{array}$$

and again the intermediate subquotients are nonunitary (but with a highest weight vector). In the action corresponding to λ, 1 is a cyclic vector for $\lambda \geqslant 2n$, hence by analytic continuation this is so for all values where the invariant form is analytic, i.e. the only exceptional points are the ones indicated. Note that for λ a nonunitary irreducible parameter the invariant form is positive on "large" representations of K, namely those with all l_i large. □

3. In the previous section we have found an ordering of the polynomial representations of $U(n)$ defining the composition series for the analytic continuation of the part U_λ of the holomorphic discrete series. In particular the bottom subquotient for $\lambda = 0, 1, \ldots, n-1$ was given by $[l] \leqslant \lambda$. In the tube realization the same subquotient was identified [5] via the Fourier transform on $H(n)$, all $n \times n$ Hermitian matrices, as essentially those holomorphic functions that are Fourier transforms of distributions supported on the boundary cone C_λ of positive definite matrices of rank no greater than λ. This establishes via the Cayley transform $c(z) = (z-i)(z+i)^{-1}$ from $\mathbf{T} = \{x + iy \mid x^* = x, y > 0\}$ onto $\mathbf{D}$.

PROPOSITION 9. *Let*

$$T^l_{ij}(z) = \det(z+i)^{-\lambda} D^l_{ij}\big((z-i)(z+i)^{-1}\big)$$

where $\lambda \in \{0, 1, \ldots, n-1\}$. Then if $[l] \leqslant \lambda$ we have that the Fourier transform of T^l_{ij} is supported by C_λ. Conversely, if $T(z)$ is a K-finite function in $H(\lambda)$ (as realized in [5]*) then T belongs to the linear span of D^l_{ij}, $[l] \leqslant \lambda$.*

In particular, for the "smallest" representation $H(1)$ of G we get that the K-finite vectors correspond to symmetric powers of the defining representation of $U(n)$ (in agreement with results of [3]) or alternatively to functions whose fourier transform is supported on C_1, positive matrices of rank 1. We conjecture that the composition series computed above can be expressed in terms of increasing rank of the Fourier transforms of the function T^l_{ij}. Specifically, it is natural to expect that it corresponds to the filtration $C_\lambda \subseteq C_{\lambda+1} \subseteq \cdots \subseteq C_n$ in the dual of $H(n)$. This again would connect the solution spaces of certain differential equations on $H(n)$ and $U(n)$.

It would no doubt be possible to extend these results to Hermitian symmetric domains for the other classical groups and compare the harmonic analysis over the noncompact flat Shilov boundary with that over the compact curved Shilov boundary. The flat boundary being homogeneous for a maximal parabolic subgroup, this pertains to the connection between the K-spectrum and the N-spectrum of the modules involved.

As in **[8]** one could also compute composition series for holomorphic vector bundles by tensoring the above results with finite-dimensional representations. Apparently there is no immediate way of generalizing the binomial formula here to operator-valued reproducing kernels, although this would be (perhaps even combinatorically) interesting.

Thanks are due to the referee for pointing out that of some relevance to the comparison of the K-spectrum with the N-spectrum is the article **[4]** of M. Kashiwara and M. Vergne, and for quoting the article **[8]** by N. Wallach, who gives a thorough and general treatment from a point of view similar to the above.

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